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The Semigroup Characterization of Osterwalder–Schrader Path Spaces and the Construction of Euclidean Fields

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Axioms are given for relativistic quantum fields so the corresponding Schwinger functions are the expectation values of Euclidean fields. The main ingredient is the characterization of Osterwalder-Schrader path spaces by the associated semigroup structure.

INTRODUCTION

Euclidean fields came of age with Nelson [10, 11, 12]. Following earlier work of Symanzik [18, 19], Nelson introduced Euclidean Markov fields and from them reconstructed relativistic quantum fields obeying the Gårding–Wightman axioms [19, 16]. Inspired by Nelson's work, Osterwalder and Schrader [13] obtained a Euclidean formulation of Quantum Field Theory, in the form of the Osterwalder–Schrader axioms for the Schwinger functions (the analytic continuation of the Wightman functions to the Euclidean region). Unlike the Wightman functions which are the expectation values of relativistic quantum fields, Schwinger functions are not in general the expectation values of Euclidean fields (Simon [15, 16] has constructed a counterexample in one space-time dimension). The existence of Euclidean fields requires that the Nelson–Symanzik positivity condition be added to the Osterwalder–Schrader axioms. As most of the recent progress in Constructive Quantum Field Theory (e.g. the Erice Lectures [1], Simon's book [16], the Marseille Proceedings [8]) has explored the structure of Euclidean fields, it is relevant to ask the question of what must be added to the Gårding–Wightman axioms so the corresponding Schwinger functions are the expectation values of Euclidean fields.

This question was first considered by Simon [15, 16]. Apart from technicalities (existence of time zero fields), Simon answered the question for Euclidean Markov fields. Unfortunately, the Markov property is conspicuously absent

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from the Osterwalder–Schrader axioms; in its place is the weaker Osterwalder–Schrader positivity condition, which replaces the Markov property in the reconstruction of relativistic quantum fields. Thus Simon had to impose a strong restriction on the relativistic field theory, namely he required cyclicity of the vacuum for the time zero fields. The same restrictions appear in Fröhlich [2, 3]. At present, the Markov property has not been verified for any of the nontrivial models ($(P(\phi)_2$, $(\phi^4)_3)$ constructed so far. These models do obey Osterwalder–Schrader positivity, as they must.

In this article we give the answer to the question of when Euclidean fields exist (again given the existence of time zero fields). The existence of Euclidean fields is shown to be equivalent to the following positivity condition:

For any positive operators $F_1, \ldots, F_n$ in the von Neumann algebra $\mathfrak{A}$ generated by the time zero fields and $t_1, \ldots, t_n \geqslant 0$,

$$\langle \Omega, e^{-t_1 H} F_1 e^{-t_2 H} F_2 \cdots e^{-t_n H} F_n \Omega \rangle \geqslant 0.$$

As the corresponding Euclidean field is Markov if and only if the vacuum Ω is cyclic for the time zero fields, and in this case the above positivity condition is equivalent to $\langle F_1 \Omega, e^{-tH} F_2 \Omega \rangle \geqslant 0$ for any positive operators $F_1, F_2 \in \mathfrak{A}$ and $t \geqslant 0$, which is the positivity condition used by Simon [15], we can see why Osterwalder–Schrader positivity is the correct condition for Euclidean fields, and in general one should not expect the Markov property. Osterwalder–Schrader positivity corresponds to cyclicity of the vacuum for the fields at *all* times, the Markov property to cyclicity for the fields at *one* time.

The main ingredient in our construction of Euclidean fields is the characterization of Osterwalder–Schrader path spaces by the associated semigroup structure (Theorem 2.4), on the lines of the characterization of Markov path spaces by positivity preserving semigroups (Simon [15], Klein and Landau [7], see also Part I, section 3). Euclidean fields have the structure of a generalized path space, and the Osterwalder–Schrader axioms requires that they satisfy Osterwalder–Schrader positivity. We perform a detailed analysis of Osterwalder Schrader path spaces. We also characterize those Osterwalder–Schrader path spaces that satisfy the Markov property. In the semigroup characterization Osterwalder–Schrader path spaces are seen to be a natural generalization of Markov path spaces.

The organization of this article is as follows: Part I has a detailed study of path spaces and the associated semigroup structures. Part II is devoted to the construction of Euclidean fields. We present axioms for relativistic quantum fields from which we construct Euclidean fields satisfying Fröhlich's axioms [3]; conversely our axioms are satisfied by relativistic quantum fields reconstructed from Euclidean fields obeying Fröhlich's axioms.

Some of these results were announced in [5, 6].

I. Path Spaces and Semigroup Structures

1. *Osterwalder–Schrader Path Spaces*

Definition 1.1. A (generalized) path space $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ consists of:

(i) a probability space (Q, Σ, μ);

(ii) a distinguished sub-σ-algebra Σ_0;

(iii) a one-parameter group $U(t)$ of measure preserving automorphisms of $L_\infty(Q, \Sigma, \mu)$ which are strongly continuous in measure;

(iv) a measure preserving automorphism R of $L_\infty(Q, \Sigma, \mu)$ such that $R^2 = I$, $RU(t) = U(-t)R$, and $RE_0 = E_0R$ where E_0 is the conditional expectation with respect to Σ_0;

(v) Σ is generated by $\bigcup_{t\in\mathbb{R}} \Sigma_t$, where $\Sigma_t = U(t)\Sigma_0$.

$E_+(E_-)$ will denote the conditional expectation with respect to $\Sigma_+(\Sigma_-)$, the σ-algebra generated by $\bigcup_{t\geqslant 0} \Sigma_t(\bigcup_{t\leqslant 0} \Sigma_t)$.

Remark 1.2. It follows $U(t)$ is a strongly continuous one-parameter group of isometries on $L_p(Q, \Sigma, \mu)$, $1 \leqslant p < \infty$ (e.g. Klein and Landau [7]), and $RE_+ = E_-R$.

Definition 1.3. An Osterwalder–Schrader path space is a path space satisfying the Osterwalder–Schrader positivity condition: $E_+RE_+ \geqslant 0$ as an operator on $L_2(Q, \Sigma, \mu)$, i.e. $\langle RF, F\rangle \geqslant 0$ for every $F \in L_2(Q, \Sigma_+, \mu)$.

Definition 1.4. A Markov path space is a path space satisfying:

(i) the reflection property: $RE_0 = E_0$

(ii) the Markov property: $E_+E_- = E_+E_0E_-$.

Proposition 1.5. (Nelson in [1]). *Every Markov path space is a Osterwalder–Schrader path space.*

Proof. By the Markov and the reflection properties, $E_+RE_+ = E_+E_-RE_+ = E_+E_0E_-RE_+ = E_+E_0RE_+ = E_+E_0E_+ = E_0 \geqslant 0$. ∎

The converse is not true, we only have

Proposition 1.6. *An Osterwalder–Schrader path space satisfies the reflection property.*

Proof. By the Osterwalder–Schrader positivity condition,

$$RE_0 = E_0RE_0 = E_0E_+RE_+E_0 \geqslant 0.$$

But R is unitary with $R^2 = I$, so R is self-adjoint and its spectrum is contained in $\{-1, 1\}$. As E_0 commutes with R and $RE_0 \geqslant 0$, it follows $RE_0 = E_0$. ∎

To every Osterwalder–Schrader path space it is associated a semigroup.

THEOREM 1.7. (Osterwalder and Schrader [13]). *Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be an Osterwalder–Schrader path space. There exists a Hilbert space $\mathscr{H}$ and a contraction $\mathscr{V}: L_2(Q, \Sigma_+, \mu) \to \mathscr{H}$ such that*

(i) *the range of $\mathscr{V}$ is dense in $\mathscr{H}$;*

(ii) *$P(t)\mathscr{V}(F) = \mathscr{V}(U(t)F)$ for $F \in L_2(Q, \Sigma_+, \mu)$ and $t \geqslant 0$ defines a strongly continuous self-adjoint contraction semigroup on $\mathscr{H}$;*

(iii) *let $\Omega = \mathscr{V}(1)$; then $\|\Omega\| = 1$ and $P(t)\Omega = \Omega$ for all $t \geqslant 0$.*

Proof. Define $\langle F \mid G\rangle = \langle RF, G\rangle$ for $F, G \in L_2(Q, \Sigma_+, \mu)$. By the Osterwalder–Schrader positivity condition $\langle\cdot \mid \cdot\rangle$ is a positive semidefinite inner product on $L_2(Q, \Sigma_+, \mu)$. Let $N = \{F \in L_2(Q, \Sigma_+, \mu) \mid \langle F \mid F\rangle = 0\}$, and take $\mathscr{H}$ to be the completion of $L_2(Q, \Sigma_+, \mu)/N$ in the positive definite inner product induced by $\langle\cdot \mid \cdot\rangle$. Let $\mathscr{V}$ be the canonical projection of $L_2(Q, \Sigma_+, \mu)$ into $\mathscr{H}$. The range of $\mathscr{V}$, Ran $\mathscr{V}$, is obviously dense in $\mathscr{H}$, and $\|\mathscr{V}(F)\|_{\mathscr{H}} = \langle F \mid F\rangle^{1/2} = \langle RF, F\rangle^{1/2} \leqslant \|F\|_2$.

Let $P(t)\mathscr{V}(F) = \mathscr{V}(U(t)F)$ for $F \in L_2(Q, \Sigma_+, \mu)$ and $t \geqslant 0$. $P(t)$ is well defined on Ran $\mathscr{V}$, as $\langle U(t)F \mid U(t)F\rangle = \langle RU(t)F, U(t)F\rangle = \langle U(-t)RF, U(t)F\rangle = \langle RF, U(2t)F\rangle = \langle F \mid U(2t)F\rangle \leqslant \langle F \mid F\rangle^{1/2} \langle U(2t)F \mid U(2t)F\rangle^{1/2}$, so $U(t)N \subset N$ for $t \geqslant 0$. $P(t)$ is clearly a strongly continuous self-adjoint semigroup on Ran $\mathscr{V}$. Moreover $\|P(t)\mathscr{V}(F)\|_{\mathscr{H}} = \|\mathscr{V}(U(t)F)\|_{\mathscr{H}} \leqslant \|U(t)F\|_2 = \|F\|_2$ for all $F \in L_2(Q, \Sigma_+, \mu)$ and $t \geqslant 0$. That $P(t)$ is a contraction then follows from

LEMMA 1.8. (Osterwalder and Schrader [13]). *Let P be a self-adjoint operator on the inner product space $\mathscr{K}$, and suppose that for every $w \in \mathscr{K}$ there exists a constant $M = M(w)$ such that*

$$\|P^n w\| \leqslant M \qquad \text{for all} \quad n = 1, 2, 3, \ldots$$

Then $\|P\| \leqslant 1$.

Proof. $\|P^n w\| = \langle w, P^{2n} w\rangle^{1/2} \leqslant \|w\|^{1/2} \|P^{2n} w\|^{1/2}$. Thus

$$\begin{aligned}\|Pw\| &\leqslant \|w\|^{1/2} \|P^2 w\|^{1/4} \leqslant \cdots \leqslant \|w\|^{\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2^n}} \|P^{2^n} w\|^{\frac{1}{2^n}} \\ &\leqslant \|w\| M^{\frac{1}{2^n}} \to \|w\| \qquad \text{as} \quad n \to \infty. \quad ∎\end{aligned}$$

This finishes the proof of the theorem, as (iii) is obvious. ∎

2. *Positive Semigroup Structures*

DEFINITION 2.1. A positive semigroup structure $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ consists of

(i) a Hilbert space $\mathscr{H}$;

(ii) a strongly continuous self-adjoint contraction semigroup $P(t)$ on $\mathscr{H}$;

(iii) a commutative von Neumann algebra $\mathfrak{A}$ of operators on $\mathscr{H}$;

(iv) a unit vector $\Omega \in \mathscr{H}$;

such that

(v) $P(t)\Omega = \Omega$ for all $t \geqslant 0$;

(vi) Ω is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{P(t) \mid t \geqslant 0\}$, i.e. the linear span of $\{P(t_2) f_2 P(t_2) \dots P(t_n) f_n \Omega \mid f_1, \dots, f_n \in \mathfrak{A}, t_1, \dots, t_n \geqslant 0\}$ is dense in $\mathscr{H}$;

(vii) for all $f_1, \dots, f_n \in \mathfrak{A}^+ = \{f \in \mathfrak{A} \mid f \geqslant 0\}$ and $t_1, \dots, t_n \geqslant 0$,

$$\langle \Omega, P(t_1) f_1 P(t_2) \dots P(t_n) f_n \Omega \rangle \geqslant 0.$$

Osterwalder–Schrader path spaces are determined by positive semigroup structures. In order to prove that, we need

LEMMA 2.2. *Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be a Osterwalder–Schrader path space, and let $\mathscr{H}$, $\mathscr{V}$, $P(t)$, Ω be as in Theorem* 1.7. *Then*

(i) *for $f \in L_\infty(Q, \Sigma_0, \mu)$, $\tilde{f}\mathscr{V}(F) = \mathscr{V}(fF)$ for $F \in L_2(Q, \Sigma_+, \mu)$ defines a bounded operator on $\mathscr{H}$ with $\|\tilde{f}\| = \|f\|_\infty$*;

(ii) *$\mathfrak{A} = \{\tilde{f} \mid f \in L_\infty(Q, \Sigma_0, \mu)\}$ is a commutative von Neumann algebra of operators on $\mathscr{H}$, with Ω as a separating vector*;

(iii) *for any $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $f_{t_i} = U(t_i) f_i$ where $f_i \in L_\infty(Q, \Sigma_0, \mu)$ and $i = 1, 2, \dots, n$,*

$$\int f_{t_1} \cdots f_{t_n} \, d\mu = \langle \Omega, \tilde{f}_1 P(t_2 - t_1) \tilde{f}_2 \cdots P(t_n - t_{n-1}) \tilde{f}_n \Omega \rangle. \qquad (*)$$

Proof. Let $f \in L_\infty(Q, \Sigma_0, \mu)$, and let

$$\tilde{f}\mathscr{V}(F) = \mathscr{V}(fF) \qquad \text{for} \quad F \in L_2(Q, \Sigma_+, \mu).$$

We identify the operator multiplication by f with the function f. As $RE_0 = E_0$ (Proposition 1.6), R is an automorphism of $L_\infty(Q, \Sigma, \mu)$, and E_0 is the conditional expectation with respect to Σ_0, it follows $fR = Rf$. Thus

$$\begin{aligned} \langle fF \mid fF \rangle &= \langle RfF, fF \rangle = \langle fRF, fF \rangle = \langle RF, |f|^2 F \rangle \\ &= \langle F \mid |f|^2 F \rangle \leqslant \langle F \mid F \rangle^{\frac{1}{2}} \langle |f|^2 F \mid |f|^2 F \rangle, \end{aligned}$$

so $fN \subset N$ and $\tilde{f}$ is well defined on $\operatorname{Ran} \mathscr{V}$. In addition, the same argument shows that

$$\|\tilde{f}^n \mathscr{V}(F)\| = \langle F \mid |f|^{2n} F \rangle^{\frac{1}{2}} \leqslant \|f\|_\infty^n \|F\|_2,$$

for $n = 1, 2, \dots$. Lemma 1.8 thus gives $\|\tilde{f}\| \leqslant \|f\|_\infty$.

On the other hand, the restriction V of $\mathscr{V}$ to $L_2(Q, \Sigma_0, \mu)$ is unitary onto its range, as $\langle G \mid F \rangle = \langle G, F \rangle$ for $G, F \in L_2(Q, \Sigma_0, \mu)$, and $\tilde{f} = VfV^*$ on the range of V. It follows $\|\tilde{f}\| \geqslant \|f\|_\infty$ so $\|\tilde{f}\| = \|f\|_\infty$. Thus the restriction of $\mathfrak{A}$ to the range of V is a von Neumann algebra with Ω as a separating vector, so it follows $\mathfrak{A}$ is a von Neumann algebra of operators on $\mathscr{H}$. It is clearly commutative.

Let $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $f_{t_i} = U(t_i) f_i$ where $f_i \in L_\infty(Q, \Sigma_0, \mu)$ for $i = 1, 2, \ldots, n$. (*) then follows from

$$f_{t_1} f_{t_2} \cdots f_{t_n} = U(t_1) f_1 U(t_2 - t_1) f_2 \cdots U(t_n - t_{n-1}) f_n 1,$$

and the fact that

$$\mathscr{V}(U(s_1) g_1 U(s_2) \cdots g_n 1) = P(s_1) \tilde{g}_1 P(s_2) \cdots \tilde{g}_n \Omega$$

for $g_1, g_2, \ldots, g_n \in L_\infty(Q, \Sigma_0, \mu)$ and $s_1, s_2, \ldots, s_n \geqslant 0$. ∎

Definition 2.3. Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be an Osterwalder–Schrader path space, and $\mathscr{H}$, $P(t)$, $\mathfrak{A}$, Ω as constructed in Theorem 1.7 and Lemma 2.2. We call $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ the associated semigroup structure.

Theorem 2.4. *Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be an Osterwalder–Schrader path space and $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ the associated semigroup structure. Then $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ form a positive semigroup structure.*

Conversely, let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure. Then there exists an Osterwalder–Schrader path space such that $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ is its associated semigroup structure.

Proof. Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be an Osterwalder–Schrader path space and $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ the associated semigroup structure. Conditions (i)–(v) of definition 2.1 are clearly satisfied; (vi) follows from the fact that 1 is a cyclic vector for $L_\infty(Q, \Sigma_0, \mu) \cup \{U(t) \mid t \geqslant 0\}$ in $L_2(Q, \Sigma_+, \mu)$, as Σ_+ is generated by $\bigcup_{t \geqslant 0} U(t)\Sigma_0$; (vii) is obvious from (*). Thus $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ form a positive semigroup structure.

Conversely, let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure. As $\mathfrak{A}$ is a commutative von Neumann algebra, $\mathfrak{A} \simeq C(Q_0)$, where Q_0, the spectrum of $\mathfrak{A}$, is a Stonean space (i.e. a compact Hausdorff totally disconnected space; e.g. Sakai [14]). Let $Q = \times_{t \in \mathbb{R}} Q_t$, where each Q_t is a copy of Q_0. Q is a compact Hausdorff space (actually a Stonean space), the "path space"; $q = (q(t))_{t \in \mathbb{R}} \in Q$ is a "path". We define $U(t)$ and R on the paths by

$$(U(t)q)(s) = q(s - t), \quad (Rq)(s) = q(-s),$$

and for any function F on Q, $(U(t)F)(q) = F(U(-t)q)$, $(RF)(q) = F(Rq)$. Clearly $U(t)$ is a one-parameter group of automorphisms of $C(Q)$ and R is an automorphism of $C(Q)$ such that $R^2 = 1$ and $RU(t) = U(-t)R$. We identify $\{F \in C(Q) \mid F(q) = f(q(0)) \text{ for } f \in C(Q_0)\}$ with $C(Q_0)$. We take Σ_0 to be σ-algebra generated by $C(Q_0)$, Σ the σ-algebra generated by $\bigcup_{t\in\mathbb{R}} \Sigma_t$, where $\Sigma_t = U(t)\Sigma_0$, and Σ_B to be the Baire σ-algebra on Q, *ie*, the σ-algebra generated by $C(Q)$. Clearly $\Sigma_0 \subset \Sigma \subset \Sigma_B$. To finish our construction we need a measure μ on (Q, Σ) such that $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ is a path space. We will construct μ as a Baire measure (i.e. on Σ_B) and then restrict it to Σ. By the Riesz–Markov theorem it suffices to construct a positive linear functional on $C(Q)$. To do so we use (*). Let us first recall that as Q_0 is a Stonean space, finite linear combination of elements of $\Xi = \{\chi \in C(Q_0) \mid \chi \text{ is a characteristic function}\}$ are dense in $C(Q_0)$. Let $\mathscr{L}(Q)$ be the set of linear combinations of functions of the form

$$F(q) = \prod_{i=1}^{n} \chi_i(q(t_i)), \qquad \text{where} \quad \chi_i \in \Xi \quad \text{for} \quad i = 1,..., n,$$

and $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$; i.e. $F = \prod_{i=1}^{n} \chi_{t_i}$ where $\chi_{t_i} = U(t_i)\chi_i$. By the Stone–Weierstrass theorem, $\mathscr{L}(Q)$ is dense on $C(Q)$. For F as above we define $\rho(F)$ by the right-hand side of (*), i.e.

$$\rho\left(\prod_{i=1}^{n} \chi_{t_i}\right) = \langle \Omega, \tilde{\chi}_1 P(t_2 - t_1)\, \tilde{\chi}_2 \cdots P(t_n - t_{n-1})\, \tilde{\chi}_n \Omega \rangle, \tag{**}$$

where $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ and $f \to \tilde{f}$ denotes the isomorphism of $C(Q_0)$ onto $\mathfrak{A}$. We extend ρ to $\mathscr{L}(Q)$ by linearity. We claim ρ is a positive linear functional on $\mathscr{L}(Q)$ (and thus well defined). Let $F \in \mathscr{L}(Q)$, F can always be written as

$$F(q) = \sum_{i=1}^{k} c_i \chi_1^i(q(t_1))\, \chi_2^i(q(t_2)) \cdots \chi_n^i(q(t_n))$$

for some $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $c_i \in \mathbb{C}$, $\chi_j^i \in \Xi$. As the χ_j^i are characteristic functions, we can choose them such that

$$(\chi_1^i(q(t_1)) \cdots \chi_n^i(q(t_n)))\, (\chi_1^j(q(t_1)) \cdots \chi_n^j(q(t_n))) = 0$$

if $i \neq j$ (for given $\chi_1, ..., \chi_n \in \Xi$, we can always choose $\chi_1', ..., \chi_m' \in \Xi$ such that $\chi_i'\chi_j' = 0$ for $i \neq j$ and each χ_i is a linear combination of the χ_j'). But then $F(q) \geqslant 0$ for all $q \in Q$ implies $c_i \geqslant 0$ for all $i = 1,..., k$, and (**) and definition 2.1(vii) show $\rho(F) \geqslant 0$. Thus ρ is a positive linear functional on $\mathscr{L}(Q)$ and can be uniquely extended by continuity to $C(Q)$. There exists then a unique Baire measure μ on Q such that

$$\rho(F) = \int F\, d\mu \qquad \text{for all} \quad F \in \mathscr{L}(Q).$$

It follows $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ is a path space. That $U(t)$ and R are measure preserving transformations follows from (**). As $P(t)$ is strongly continuous on $\mathscr{H}$ it follows from (**) that $U(t)$ is weakly continuous on $L_2(Q, \Sigma, \mu)$. As $U(t)$ is unitary, it is strongly continuous. This implies strong continuity in measure.

Let us prove $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ satisfies Osterwalder–Schrader positivity. It suffices to show $\langle RF, F\rangle \geqslant 0$ for

$$F = \sum_{i=1}^{k} c_i \chi_{t_1}^i \cdots \chi_{t_n}^i, \ 0 \leqslant t_1 \leqslant \cdots \leqslant t_n, \ \chi_{t_i} = U(t_i)\, \chi_i, \ \chi_i \in \Xi.$$

But

$$F = \sum C_i U(t_1)\, \chi_1{}^i U(t_2 - t_1) \cdots U(t_n - t_{n-1})\, \chi_n{}^i 1$$

and

$$\begin{aligned}
&\langle RF, F\rangle \\
&= \sum_{i,j} c_i{}^* c_j \langle U(-t_1)\, \chi_1{}^i U(-(t_2 - t_1))\, \chi_2{}^i \cdots \chi_n{}^i 1,\ U(t_1)\, \chi_1{}^j U(t_2 - t_1)\, \chi_2{}^j \cdots \chi_n{}^j 1\rangle \\
&= \sum_{i,j} c_i{}^* c_j \langle 1,\ \chi_n^{i*} U(t_n - t_{n-1})\, \chi_{n-1}^{i*} \cdots \chi_1^{i*} U(t_1)\, U(t_1)\, \chi_1{}^j U(t_2 - t_1) \cdots \chi_n{}^j 1\rangle \\
&= \sum_{i,j} c_i{}^* c_j \langle \Omega,\ \tilde{\chi}_n^{i*} P(t_n - t_{n-1})\, \tilde{\chi}_{n-1}^{i*} \cdots \tilde{\chi}_1^{i*} P(t_1)\, P(t_1)\, \tilde{\chi}_1{}^j P(t_2 - t_1) \cdots \tilde{\chi}_n{}^j \Omega\rangle \\
&= \sum_{i,j} c_i{}^* c_j \langle P(t_1)\, \tilde{\chi}_1{}^i P(t_2 - t_1)\, \tilde{\chi}_2{}^i \cdots \tilde{\chi}_n{}^i \Omega,\ P(t_1)\, \tilde{\chi}_1{}^j P(t_2 - t_1)\, \tilde{\chi}_2{}^j \cdots \tilde{\chi}_n{}^j \Omega\rangle \\
&= \Big\langle \sum_i c_i P(t_1)\, \tilde{\chi}_1{}^i P(t_2 - t_1)\, \tilde{\chi}_2{}^i \cdots \tilde{\chi}_n{}^i \Omega,\ \sum_j c_j P(t_1)\, \tilde{\chi}_1{}^j P(t_2 - t_1)\, \tilde{\chi}_2{}^j \cdots \tilde{\chi}_n{}^j \Omega \Big\rangle \\
&\geqslant 0.
\end{aligned}$$

Thus $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ is a Osterwalder–Schrader path space. Let $(\mathscr{H}', P(t)', \mathfrak{A}', \Omega')$ be the associated semigroup structure. Then the linear map $T\colon \mathscr{H}' \to \mathscr{H}$ defined by

$$T(P(t_1)'\, \tilde{\chi}_1{}' P(t_2)' \cdots \tilde{\chi}_n{}'\Omega') = P(t_1)\, \tilde{\chi}_1 P(t_2) \cdots \tilde{\chi}_n \Omega$$

is clearly unitary by (*) and (**) (and thus well defined) and preserves the positive semigroup structure, so we can identify $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ with the associated semigroup structure. ∎

Remark 2.5. Let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure. It follows from Theorem 2.4 and Lemma 2.2 that Ω is separating for $\mathfrak{A}$. It also follows that if $\mathfrak{A}$ is only supposed to be a commutative C^*-algebra whose spectrum is a Stonean space then $\mathfrak{A}$ is automatically a von Neumann algebra.

3. *Markov Path Spaces*

We now characterize those Osterwalder–Schrader path spaces that are Markov.

THEOREM 3.1. *Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be a Osterwalder–Schrader path space, and $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ the associated semigroup structure. $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ is Markov if and only if Ω is a cyclic vector for $\mathfrak{A}$.*

Remark 3.2. Theorems 2.4 and 3.1 tell us that Markov path spaces correspond to positive semigroup structures in which condition (vi) of Definition 2.1 is replaced by the stronger.

(vi)′ Ω is a cyclic vector for $\mathfrak{A}$.

Thus in the semigroup structure characterization Osterwalder–Schrader path spaces are a natural generalization of Markov path spaces.

Remark 3.3. For a positive semigroup structure corresponding to a Markov path space (i.e. satisfying Definition 2.1 with (vi)′ substituted for (vi)), (vii) is equivalent to

(vii) for all $f, g \in \mathfrak{A}^+ = \{f \in \mathfrak{A} \mid f \geqslant 0\}$ and $t \geqslant 0$, $\langle f\Omega, P(t)g\Omega\rangle \geqslant 0$.
This is the positivity condition used by Simon [15].

Before we prove the theorem, let us note

DEFINITION 3.4. Let M be a probability space. A strongly continuous self-adjoint contraction semigroup $P(t)$ on $L_2(M)$ is said to be positivity preserving if

(i) $P(t)1 = 1$ for all $t \geqslant 0$;
(ii) $P(t)f \geqslant 0$ for all $f \in L_2(M)$, $f \geqslant 0$.

PROPOSITION 3.5. *Let $P(t)$ be a positivity preserving semigroup on $L_2(M)$. Then $(L_2(M), P(t), L_\infty(M), 1)$ form a positive semigroup structure, with 1 a cyclic vector for $L_\infty(M)$.*

Conversely, let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure, with Ω a cyclic vector for $\mathfrak{A}$. There exists a probability space M and a positivity preserving semigroup $P_1(t)$ on $L_2(M)$ such that $(\mathscr{H}, P(t), \mathfrak{A}, \Omega) \cong (L_2(M), P_1(t), L_\infty(M), 1)$ as positive semigroup structures.

Remark 3.6. Thus Theorems 2.4, 3.1, and Proposition 3.5 show that Markov path spaces are determined by positivity preserving semigroups (see Simon [15], Klein and Landau [7]).

Proof of Proposition 3.5. We only have to prove the converse. Let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure with Ω a cyclic vector for $\mathfrak{A}$. It follows $\mathfrak{A}$ is maximal abelian, and there exists a Baire measure ν on the spectrum Q_0 of $\mathfrak{A}$ such that $\mathscr{H} \cong L_2(Q_0, \nu)$, $\mathfrak{A} \cong L_\infty(Q_0, \nu)$ and $\Omega \cong 1$. If $P_1(t)$ corresponds to $P(t)$ under the above isomorphism, $P_1(t)$ is clearly a positivity preserving semigroup on $L_2(Q, \nu)$. ∎

Proof of Theorem 3.1. If the path space is Markov, $E_+RE_+ = E_0$ (see proof of Proposition 1.5), so

$$\langle RF, F\rangle = \langle E_0F, E_0F\rangle \qquad \text{for all} \quad F \in L_2(Q, \Sigma_+, \mu),$$

and thus $\mathscr{H} \simeq L_2(Q, \Sigma_0, \mu)$, $\mathfrak{A} \simeq L_\infty(Q, \Sigma_0, \mu)$, $P(t) \simeq E_0U(t)E_0$ and $\Omega \simeq 1$, so Ω is cyclic for $\mathfrak{A}$.

Conversely, if Ω is cyclic for $\mathfrak{A}$, $P(t)$ is a positivity preserving semigroup (Proposition 3.5), so it follows $P(t)\,\mathfrak{A}\Omega \subset \mathfrak{A}\Omega$ for all $t \geqslant 0$ (e.g. Klein and Landau [7]). The Markov property then follows as in [15, 7]: by an explicit computation, using (*), it follows that if $F(q) = f_n(q(t_n)) \cdots f_1(q(t_1))$, where $f_1, ..., f_n \in L_\infty(Q, \Sigma_0, \mu)$, and $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant 0$, then $E_+E_-F = E_+F = P(-t_n)\tilde{f}_n \cdots \tilde{f}_2P(t_2 - t_1)\tilde{f}_1\Omega \in \mathfrak{A}\Omega \simeq L_\infty(Q, \Sigma_0, \mu)$ and is thus measurable with respect to Σ_0. Hence $E_+E_- = E_+E_0E_-$. ∎

II. Euclidean Fields

We consider a scalar Bose field φ in $n + 1$ space-time dimensions. Points in $\mathbb{R}^{n+1}$ will be denoted by (x, t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. $\mathscr{S}(\mathbb{R}^d)$ will denote the real Schwartz space over $\mathbb{R}^d$.

Axiom 0. φ is a relativistic scalar Bose field in $n + 1$ space-time dimensions satisfying the Garding–Wightman axioms, except maybe for the uniqueness of the vacuum. $\mathscr{H}$ denotes the Hilbert space, H the Hamiltonian (positive self-adjoint), Ω the vacuum.

Axiom I. The relativistic fields at fixed times exist and determine the field theory:

I.1. The time zero fields $\varphi(h, 0) \equiv \varphi(h \otimes \delta_0)$ exist and are self-adjoint for all $h \in \mathscr{S}(\mathbb{R}^n)$, with the vacuum Ω in the domain of $\varphi(h, 0)$ and $\langle\Omega, \varphi(h, 0)^2\Omega\rangle$ continuous on $\mathscr{S}(\mathbb{R}^n)$, and the operators $\{e^{i\varphi(h,0)} \mid h \in \mathscr{S}(\mathbb{R}^n)\}$ generate a commutative von Neumann algebra $\mathfrak{A}$ on $\mathscr{H}$.

I.2. $\pm\varphi(h, 0) \leqslant |||h||| \, (H + 1)$ on the quadratic form domain $Q(H)$ of H for some continuous norm $|||\cdot|||$ on $\mathscr{S}(\mathbb{R}^n)$.

I.3. Let $\varphi(h, t) = e^{itH}\varphi(h, 0)e^{-itH} \equiv \varphi(h \otimes \delta_t)$; the vacuum Ω is a cyclic vector for the von Neumann algebra generated by the operators $\{e^{i\varphi(h,t)} \mid h \in \mathscr{S}(\mathbb{R}^n), t \in \mathbb{R}\}$ on $\mathscr{H}$.

I.4. $\varphi(f) = \int \varphi(f_t, t)\, dt$ as a quadratic form on $Q(H)$ for all $f \in \mathscr{S}(\mathbb{R}^{n+1})$, where $f_t(x) = f(x, t)$.

AXIOM II. Positivity condition:

For all positive operators $F_1, ..., F_n \in \mathfrak{A}$ and $t_1, ..., t_n \geqslant 0$,

$$\langle \Omega, e^{-t_1 H} F_1 e^{-t_2 H} F_2 \cdots e^{-t_n H} F_n \Omega \rangle \geqslant 0.$$

We will also consider Axiom I′, which consists of Axiom I with I.2 replaced by the stronger:

I.2′. $\pm\varphi(h, 0) \leqslant K \| h \|_1 (H + 1)$ on the quadratic form domain $Q(H)$ of H for some constant K, where $\| h \|_1 = \int | h(x) | \, dx$.

I.2′ is just I.2 plus the requirement on $|\!|\!| \cdot |\!|\!|$ that $|\!|\!| h |\!|\!| \leqslant K \| h \|_1$.

Remarks. 1. The technical conditions I.1, I.2 were used by Fröhlich [2, 3] (see also Simon [15, 16]). The justification of I.4 is due to Nelson [9, 11], who shows also that $\varphi(f)$ is essentially self-adjoint on $C^\infty(H)$ and leaves it invariant.

2. Axioms 0, I, II are satisfied by the $P(\phi_2)$ models with half-Dirichlet boundary conditions (this follows from the theorem below and the results in Fröhlich [3]). Actually $P(\phi)_2$ models satisfy the stronger Axioms 0, I′, II, as I.2′ is the original form of the Glimm–Jaffe φ-bound [4].

Axioms 0, I, II will enable us to construct a Euclidean field ϕ obeying a form of Fröhlich's axioms [3]. We briefly outline Fröhlich's axioms, in a form suitable for our purposes.

AXIOM A. Existence of a Euclidean field over $\mathscr{S}(\mathbb{R}^{n+1})$:

A.a. There exists a probability space (Q, Σ, μ) and a continuous linear map ϕ from $\mathscr{S}(\mathbb{R}^{n+1})$ into the set of random variables over (Q, Σ, μ) equipped with the topology of convergence in measure. Let $J(f) = \int e^{i\phi(f)} \, d\mu$. (A.a is equivalent to A1-4 in [3], i.e. to the existence of a functional J on $\mathscr{S}(\mathbb{R}^{n+1})$ such that $J(0) = 1$, J is continuous on $\mathscr{S}(\mathbb{R}^{n+1})$, J is of positive type, and $J(f) = J(-f)^*$).

A.b. J is time-translation and time-reflection invariant (A5 in [3]).

A.c. J is Euclidean invariant (A6 in [3]).

AXIOM B. Osterwalder–Schrader positivity.

AXIOM C. Exponential J-bound and bound on the covariance (two-point) function.

Axioms A.a–b, B, C imply that $\phi(f) \in L_p$ for all $1 \leqslant p < \infty$ and $f \in \mathscr{S}(\mathbb{R}^{n+1})$, and thus the expectation values

$$S_n(f_1, ..., f_n) = \int \phi(f_1) \cdots \phi(f_n) \, d\mu, \qquad \text{where} \quad f_1, ..., f_n \in \mathscr{S}(\mathbb{R}^{n+1}),$$

exist [3]. By the noncoincident expectation values we denote the restriction of the S_n's to functions $f_1, \ldots, f_n$ with pairwise disjoint supports. The Osterwalder–Schrader axioms [13] are expressed in terms of the noncoincident expectation values.

We will also consider the weaker:

A.c′. The noncoincident expectation values are Euclidean invariant.

By Axiom A' we will mean A.a, A.b, and A.c′. Fröhlich [3] has shown that Axioms A', B, C imply the Osterwalder–Schrader axioms.

THEOREM. *Let φ be a relativistic scalar Bose field satisfying Axioms* 0, I, II; *then the corresponding Schwinger functions are the expectation values of a Euclidean field ϕ satisfying Axioms A', B, C. Conversely, given a Euclidean field satisfying Axioms A', B, C, the reconstructed relativistic quantum field satisfies Axioms* 0, I, II.

If φ satisfies Axioms 0, I′, II, *then ϕ satisfies Axioms A, B, C.*

Remarks. 3. The corresponding Euclidean field is Markov if and only if Ω is cyclic for the time zero fields algebra $\mathfrak{A}$. In this case Axiom II is equivalent to

$$\langle F\Omega, e^{-tH}G\Omega\rangle \geqslant 0 \qquad \text{for all positive } F, \quad G \in \mathfrak{A} \text{ and } t \geqslant 0,$$

which is the positivity condition used by Simon [15].

4. We can thus see why Osterwalder–Schrader positivity is the correct condition for Euclidean fields, and in general one should not expect the Markov property, Osterwalder–Schrader positivity corresponds to cyclicity of the vacuum for the field at *all* times, the Markov property to cyclity for the field at *one* time.

5. Since I.1, I.2, and I.4 are essentially technical, and I.3 is just a restatement of the cyclicity of the vacuum, Axiom II is the basic ingredient needed to construct a Euclidean field. It is thus the relativistic translation of the Nelson–Symanzik positivity condition.

In the proof of the theorem we will use the following simple lemma.

LEMMA. *Let $\mathscr{H}$ be a Hilbert space, $\mathfrak{A}$ a von Neumann algebra of operatiors on $\mathscr{H}$, $\Omega \in \mathscr{H}$ such that $H\Omega = 0$. Then Ω is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{e^{-tH} \mid t \geqslant 0\}$ if and only if Ω is a cyclic vector for the algebra generated by $\{e^{itH}Fe^{-itH} \mid F \in \mathfrak{A}, t \in \mathbb{R}\}$.*

Proof. As $\{e^{-tH} \mid t \geqslant 0\}$ and $\{e^{itH} \mid t \in \mathbb{R}\}$ generate the same von Neumann algebra, Ω is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{e^{-tH} \mid t \geqslant 0\}$ if and only if it is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{e^{itH} \mid t \in \mathbb{R}\}$. As $e^{itH}\Omega = \Omega$ for all $t \in \mathbb{R}$, the latter is true if and only if Ω is a cyclic vector for the algebra generated by $\{e^{itH}Fe^{-itH} \mid F \in \mathfrak{A}, t \in \mathbb{R}\}$. ∎

Proof of Theorem. Let φ be a relativistic quantum field satisfying Axioms 0, I, II. It follows (with the help of the Lemma) that $(\mathscr{H}, e^{-tH}, \mathfrak{A}, \Omega)$ form a positive semigroup structure. By Theorem 2.4 there exists an Osterwalder–Schrader path space $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ such that $(\mathscr{H}, e^{-tH}, \mathfrak{A}, \Omega)$ is its associated semigroup structure. We recall $\mathfrak{A} \cong L_\infty(Q, \Sigma_0, \mu)$, so let $\phi(h, 0)$ denote the measurable function on (Q, Σ_0, μ) corresponding to $\varphi(h, 0)$, where $h \in \mathscr{S}(\mathbb{R}^n)$. Let $\phi(h, t) = U(t)\,\phi(h, 0)$. Let $|||\, h \,|||_1 = \langle \Omega, \varphi(h, 0)^2 \Omega \rangle^{\frac{1}{2}} = \| \varphi(h, 0)\Omega \|$. Then $|||\cdot|||_1$ is a continuous seminorm on $\mathscr{S}(\mathbb{R}^n)$. For $f \in \mathscr{S}(\mathbb{R}^{n+1})$, let $f_t(x) = f(x, t)$ and let

$$\| f \|_{s_1} = \int |||\, f_t \,||| \, dt.$$

$\|\cdot\|_{s_1}$ is clearly a continuous seminorm on $\mathscr{S}(\mathbb{R}^{n+1})$. Define

$$\phi(f) = \int \phi(f_t, t)\, dt$$

as a Riemann integral in $L_2(Q, \Sigma, \mu)$. It then follows by an argument of Fröhlich [2] that

$$\left| \int \phi(f)\,\phi(g)\, d\mu \right| \leqslant \| f \|_{s_1} \| g \|_{s_1} \qquad \text{for all} \quad f, g \in \mathscr{S}(\mathbb{R}^{n+1}).$$

Let $J(f) = \int e^{i\phi(f)}\, d\mu$ for $f \in \mathscr{S}(\mathbb{R}^{n+1})$, we have [2]

$$| J(f+g) - J(f) | \leqslant \int \phi(g)^2\, d\mu \leqslant \| g \|_{s_1}^2 .$$

Thus J satisfies Axiom A.a–b. Axiom B is just Osterwalder–Schrader positivity. Axiom C follows from Axiom I.1–2 as in [3]. A.c′ follows as in [13].

Let $|||\cdot|||$ be the norm in I.2, and let

$$\| f \|_s = \int |||\, f_t \,|||\, dt + \sup_t |||\, f_t \,|||$$

for $f \in \mathscr{S}(\mathbb{R}^{n+1})$. $\|\cdot\|_s$ is clearly a continuous norm. It follows [2, 3] that $J(zf)$ is analytic on the domain $\{ z \in C \mid |\operatorname{Im} z| < 1/|||\, f \,|||_s \}$ and

$$| J(zf) | < e^{|\operatorname{Im} z|\, \| f \|_s}.$$

Moreover, if $S_m(f_1, ..., f_m) = \int \phi(f_1) \cdots \phi(f_m)\, d\mu$, then

$$| S_m(f_1, ..., f_m) | \leqslant C_m \| f_1 \|_s \cdots \| f_m \|_s$$

for some constant C_m depending only on m.

Let us now assume φ satisfies II.2′, i.e., $|||\, h \,||| = K \| h \|_1$. A.c then follows as remarked by Simon [16, p. 207] (one does need though to require $|||\cdot|||$ to have a special form, like the above, because of the $\sup_t |||f_t|||$ term in $\|f\|_s$): It suffices to show the Euclidean invariance of the expectation values $S_m(f_1,\ldots,f_m)$, for $J(zf)$ analytic on the strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < 1/|||f|||_s\}$, so

$$J(zf) = \sum_{m=0}^{\infty} (z^m/m!)\, S_m(f,\ldots,f)$$

for small z, and thus the Euclidean invariance of the S_m's implies the Euclidean invariance of J. As the noncoincident S_m's are Euclidean invariant, it suffices to show that any expectation value can be approximated by noncoincident expectation values. As we have uniform bounds on the S_m's, it is enough to approximate $S_m(f_1,\ldots,f_m)$ when $f_1,\ldots,f_m$ have compact supports, say contained in a cube in $\mathbb{R}^{n+1}$ centered at zero and with sides of length M. To simplify the notation we will give the proof in $\mathbb{R}^{1+1}$, but the same proof holds in general. Let $\chi_i(x)$ be the characteristic function of the interval $[-M + (i-1)(2M/N),\ -M + i(2M/N))$, $i = 1,\ldots, N$. Then $f_j = \sum_{i=1}^{N} f_{ji}$, where $f_{ji}(x, t) = f_j(x, t)\, \chi_i(x)$, for all $j = 1,\ldots, m$. Thus $S_m(f_1,\ldots,f_m) = \sum S_m(f_{1i_1},\ldots,f_{mi_m})$, where the summation is over all i_j, $i = 1,\ldots, N$, $j = 1,\ldots, m$. Let $S_m(f_1,\ldots,f_m) = \sum' S_m(f_{1i_1},\ldots,f_{mi_m}) + \sum'' S_m(f_{1i_1},\ldots,f_{mi_m})$, where the summation $\sum'$ involves all terms in which no two of the i_j's are equal, and the other terms are collected in $\sum''$. It suffices to show $|\sum'' S_m(\cdots)| \to 0$ as $N \to \infty$. But

$$|S_m(f_{1i_1},\ldots,f_{mi_m})| \leqslant \|f_{1i_1}\|_s \cdots \|f_{mi_m}\|_s\,,$$

and $\|f_{ji}\|_s \leqslant C\|f_j\|_\infty N^{-1}$, for some constant C depending only on M. Thus $|S_m(f_{1i_1},\ldots,f_{mi_m})| \leqslant C'N^{-m}$ for some constant C' depending only on m, $M, f_1,\ldots,f_m$. Now $\sum''$ involves $N^m - N(N-1)\cdots(N-m+1)$ terms, so

$$\left|\sum{}'' S_m(\cdots)\right| \leqslant C'N^{-m}(N^m - N(N-1)\cdots(N-m+1)) \to 0$$

as $N \to \infty$. Thus Axioms 0, I′, II imply Axioms A, B, C.

The converse (i.e., that if ϕ is a Euclidean field satisfying Axioms A, B, C, then the reconstructed relativistic quantum field φ satisfies 0, I, II) follows from Theorem 1.7, Lemma 2.2, and the results in [3]. ∎

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